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On the Distance to Optimality of the Geometric Approximate Minimum-Energy Attitude Filter

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***Abstract*— This paper studies the near-optimality of the recent geometric approximate minimum-energy (GAME) filter, an attitude filter for estimation on the rotation group $\mathrm{SO}(3)$. The GAME filter approximates the minimum-energy (optimal) filtering solution by truncating the derivatives of the associated value function of order higher than 2. In this work, this approximation is pinned down in terms of an analytic expression for a bound on the difference between the cost attained by the GAME filter and the minimum-energy cost. This bound, called the optimality gap, is shown to be small in normal operation conditions. This is further supported by simulations.**

I. INTRODUCTION

Attitude filtering is a challenging problem that is widely studied in many robotics applications where a reliable attitude estimate is essential for localizing, feedback loop control and other control related tasks. Attitude denotes the rotation matrix transformation between the body-fixed frame and the reference frame of the moving object.

Classical optimal filtering results on this topic have mainly focused on utilizing the Kalman filter [1] by either linearizing the underlying nonlinear system, as in the extended Kalman filter (EKF) [2], or by sampling and approximating the associated nonlinear probability distributions, e.g. in the unscented Kalman filter [3] and the particle filter [4].

Inspired by new challenging applications such as automated control of unmanned aerial vehicles (UAV)s, more recent attitude filtering methods are exploiting the geometric structure of the problem to achieve improved estimation error performance. The multiplicative extended Kalman filter [5], a unit quaternion version of the EKF, has attracted much interest due to its superior performance in spacecraft applications, as was concluded by a relatively recent survey on attitude filtering [6]. Recently, the authors proposed the geometric approximate minimum-energy (GAME) filter [7], [8], a filter posed on the space of rotation matrices $\mathrm{SO}(3)$ that is based on minimum-energy filtering [9], [10]. The MEKF was benchmarked by the authors in [7], [8] where it was shown in simulations that the GAME filter actually achieves lower estimation error than a number of well-known geometric attitude filters including the MEKF, the invariant Kalman filter [11], [12], the right invariant Kalman filter [13], the unscented quaternion estimator [14] and the nonlinear constant gain observer [15].

Attitude filtering based on the classical stochastic or deterministic optimal filtering costs, has proven to be a difficult problem for which an infinite dimensional filter appears to be the complete solution [16]. Consequently, only approximate solutions of the problem exist in the literature except for methods based on non-classical cost functionals [17], [18]. Therefore, it is important to identify performance measures that can quantify the performance of a sub-optimal attitude filter. Recent results by Coote *et al.* [19] and the authors [20] utilized a least squares argument that yields a measure on the distance to optimality of the attitude filter. These works only considered the special cases of attitude filtering on the unit circle, i.e. a fixed rotation axis and attitude filtering on $\mathrm{SO}(3)$ using full state measurements, respectively.

In this paper, the least squares analysis in [20] is extended to the recently proposed GAME filter [7], [8], an attitude filter on $\mathrm{SO}(3)$ with vectorial measurements. We prove that the distance to optimality of the GAME filter can be computed using an analytical expression of an upper bound on the difference between the cost attained by the GAME filter and the minimum-energy cost. The upper bound, referred to as the ‘optimality gap’, is derived despite the fact that the minimum-energy filter is not explicitly known. We show that under normal operational conditions the ‘optimality gap’ is a small value that we further investigate using Monte Carlo experiments involving a range of measurement error levels and providing strong support for this claim. We show that after the transient period of the estimation, the GAME filter achieves a cost near to the minimum-energy cost and hence functioning close to the minimum-energy (optimal) filter.

The remainder of the paper is organized as follows. Section II introduces the notation and contains several useful identities that are later invoked in the derivation of the results. In Section III we introduce the attitude filtering problem posed on the group of rotation matrices $\mathrm{SO}(3)$. Section V contains the main results of the paper as well as the derivation of the optimality gap. In Section VI Monte Carlo simulations are provided that further study the claims of the paper. Finally Section VII concludes the paper.

II. PRELIMINARIES

The rotation group is denoted by $\mathrm{SO}(3)$.

$$\mathrm{SO}(3) = \{X \in \mathbb{R}^{3\times 3} \,|\, X^\top X = I, \det(X) = 1\},$$

*This research was supported by the Australian Research Council through Discovery Grant DP120100316 ‘Geometric observer theory for mechanical control systems’.

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Preprint submitted to 2014 American Control Conference.
Received September 27, 2013.

where I is the 3 by 3 identity matrix. The associated Lie algebra $\mathfrak{so}(3)$ is the set of skew-symmetric matrices,

$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} \,|\, A = -A^\top\}.$$

For $\Omega = [a, b, c]^\top \in \mathbb{R}^3$, the lower index operator $(.)_\times : \mathbb{R}^3 \longrightarrow \mathfrak{so}(3)$ yields the skew-symmetric matrix

$$\Omega_\times = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}.$$

Inversely, the operator vex : $\mathfrak{so}(3) \longrightarrow \mathbb{R}^3$ extracts the skew coordinates, $\mathrm{vex}(\Omega_\times) = \Omega$. Every rotation matrix $X \in \mathrm{SO}(3)$ can be represented using the angle axis coordinates

$$X = \exp(\theta a_\times), \tag{1}$$

where the unit vector $a \in \mathbb{R}^3$ is the axis of rotation, θ is the angle of rotation with respect to the axis and exp is the matrix exponential. The matrix exponential from $\mathfrak{so}(3)$ to $\mathrm{SO}(3)$ yields

$$X = I + \sin(\theta) a_\times + (1 - \cos(\theta)) a_\times^2. \tag{2}$$

Let $L_X : \mathrm{SO}(3) \longrightarrow \mathrm{SO}(3)$, $L_X S = XS$, be the left translation and let $TL_X : T\,\mathrm{SO}(3) \longrightarrow T\,\mathrm{SO}(3)$ denote the associated tangent map for $\Gamma \in \mathfrak{so}(3)$ and $X, S \in \mathrm{SO}(3)$. We use the standard left-invariant Riemannian metric on $\mathrm{SO}(3)$. That is, for $\Gamma, \Omega \in \mathfrak{so}(3)$ and $X \in \mathrm{SO}(3)$

$$\begin{aligned} \langle \mathrm{TL}_X \Gamma, \mathrm{TL}_X \Omega \rangle_X = \langle \Gamma, \Omega \rangle_I &:= \frac{1}{2} \mathrm{trace}(\Gamma^\top \Omega) \\ &= \mathrm{vex}(\Gamma)^\top \mathrm{vex}(\Omega). \end{aligned} \tag{3}$$

Denote a symmetric positive semi-definite matrix by $B \geq 0$ (a symmetric positive definite matrix is denoted by $B > 0$). The seminorm $\|.\|_R : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}_0^+$ is given by

$$\|M\|_R := \sqrt{\frac{1}{2} \mathrm{trace}(MRM^\top)}, \tag{4}$$

where $R \in \mathbb{R}^{3\times 3} \geq 0$. Note that if $R > 0$ then $\|M\|_R$ coincides with the Frobenius norm of $MR^{1/2}$. The symmetric projector $\mathbb{P}_s$ is defined by

$$\mathbb{P}_s(M) := 1/2(M + M^\top). \tag{5}$$

The skew-symmetric projector $\mathbb{P}_a$ is defined by

$$\mathbb{P}_a(M) := 1/2(M - M^\top). \tag{6}$$

Note that for every $A \in \mathfrak{so}(3)$, $M \in \mathbb{R}^{3\times 3}$ and $S = S^\top \in \mathbb{R}^{3\times 3}$,

$$\mathrm{trace}(A \mathbb{P}_s(M)) = 0, \ \mathrm{trace}(\mathbb{P}_s(SA)) = 0. \tag{7}$$

The following identities hold for every $\gamma, \psi \in \mathbb{R}^3$, $X \in \mathrm{SO}(3)$ and $S = S^\top \in \mathbb{R}^{3\times 3}$.

$$\gamma_\times \psi_\times = \psi \gamma^\top - \gamma^\top \psi I. \tag{8}$$

$$\psi \times \gamma = \psi_\times \gamma = 2\,\mathrm{vex}\,\mathbb{P}_a(\gamma \psi^\top) = 2\,\mathrm{vex}\,\mathbb{P}_a(\psi_\times \gamma_\times). \tag{9}$$

$$\mathbb{P}_a(S \gamma_\times) = \frac{1}{2}((\mathrm{trace}(S) I - S)\gamma)_\times. \tag{10}$$

$$\mathrm{trace}(\gamma_\times^\top S \psi_\times) = \gamma^\top (\mathrm{trace}(S) I - S)\psi. \tag{11}$$

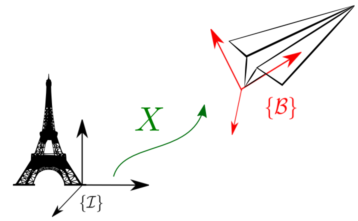

Fig. 1. Rigid-body motion frames.

$$\gamma^\top S \psi = \frac{1}{2} \mathrm{trace}(\gamma_\times^\top [\mathrm{trace}(S) I - 2S] \psi_\times). \tag{12}$$

$$(X\gamma)_\times = X \gamma_\times X^\top. \tag{13}$$

III. ATTITUDE FILTERING

In this section we introduce the problem of minimum-energy attitude filtering using the kinematics of a rigid-body traveling in $3D$ space. The minimum-energy optimization problem is carefully formulated on the space of rotation matrices belonging to the Lie group $\mathrm{SO}(3)$. $\mathbb{R}^3$ measurements are however modeled on the vector space to be consistent with the data obtained from robotic sensors such as gyros, star sensors and vision sensors.

Consider the plane shown in Figure 1, as an example of a rigid-body moving in the $3D$ space. The two coordinate frames identified in the picture are the inertial frame or the reference frame that is a known frame fixed at some reference point and is denoted by $\{\mathcal{I}\}$ and the body-fixed frame that is a moving coordinate frame fixed to the aircraft and denoted by $\{\mathcal{B}\}$. The attitude matrix X converts the coordinates of the inertial frame to the coordinates of the body-fixed frame.

The attitude kinematics is given by

$$\dot{X} = X \Omega_\times, \ X(0) = X_0, \tag{14}$$

where the attitude X is an $\mathrm{SO}(3)$-valued state signal with the unknown initial value X_0 and $\Omega \in \mathbb{R}^3$ represents the angular velocity of the moving body expressed in the body-fixed frame.

A rate-gyro sensor measures the angular velocity through the following equation

$$u = \Omega + Bv. \tag{15}$$

The signals $u \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote the body-fixed frame measured angular velocity and the input measurement error, respectively. The coefficient matrix $B \in \mathbb{R}^{3\times 3}$ allows for different weightings for the components of the unknown input measurement error v. We assume that B is full rank and hence that $Q := BB^\top$ is positive definite. Often an unknown slowly time-varying bias signal is also included in the rate-gyro measurement that is left out to simplify the analysis of the paper.

Consider the vectors $\mathring{y}_i \in \mathbb{R}^3$ as known vector directions in the reference frame. Measuring these vectors in the body-fixed frame provides partial information about the attitude X. Typically, magnetometers, visual sensors, sun sensors or star trackers are deployed for this purpose. The following model yields the measurements of theses sensors.

$$y_i = X^\top \mathring{y}_i + D_i w_i,\ i = 1, \cdots, n \tag{16}$$

The measurements $y_i \in \mathbb{R}^3$ are measurements of the $\mathring{y}_i$ in the body-fixed frame and the signals $w_i \in \mathbb{R}^3$ are the unknown output measurement errors. The coefficient matrix $D_i \in \mathbb{R}^{3\times 3}$ allows for different weightings of the components of the output measurement error w_i. Again we assume that D_i is full rank and $R_i := D_i D_i^\top$ is positive definite.

The filtering problem at each time t is that, given the measurements $\{y_i|_{[0,\ t]}\}$ and $u|_{[0,\ t]}$, the goal is to obtain a minimum-energy estimate $\hat{X}(t)$ of the true state $X(t)$ by minimizing the cost (17). The definition of a minimum-energy estimate and the methodology of our filtering is the subject of the next section.

IV. MINIMUM-ENERGY FILTERING

Consider the cost

$$J(t;\ X_0,\ \mathcal{X},\ v_\Omega|_{[0,\ t]},\ \{w_i|_{[0,\ t]}\}) = \frac{1}{2}\|X_0 - \mathcal{X}\|^2_{K_{X_0}} + \frac{1}{2}\int_0^\top \left(v_\Omega^\top v_\Omega + \sum_i w_i^\top w_i \right) d\tau, \tag{17}$$

in which $K_{X_0} \in \mathbb{R}^{3\times 3} > 0$. The initial state cost is constructed by embedding rotations in the space of 3 by 3 matrices and using (4). Note that $\mathcal{X} \in \mathrm{SO}(3)$ is an a priori value for the initial state $X(0)$ depending on the problem. If such a value is not available the identity matrix can be used for $\mathcal{X}$.

The cost (17) can be thought of as a measure of the aggregate energy stored in the unknown initial state and measurement signals of (1), (15) and (16) and therefore by minimizing (17) over these unknowns we seek an minimum-energy set of these unknowns that explain the measurement data (15) and (16) and furthermore yield the minimum-energy estimate $\hat{X}(t)$.

In order to obtain $\hat{X}(t)$, one seeks a combination of the unknowns $(X_0,\ v|_{[0,\ t]}, \Omega\ \{w_i|_{[0,\ t]}\})$ that is compatible with the measurements $\{y_i|_{[0,\ t]}\}$ and $u|_{[0,\ t]}$ in fulfilling the system equations (1). Note that in general, infinitely many combinations of these unknowns are compatible with the measurements. By minimizing the cost (17) a triplet $(X_0^*,\ v^*|_{[0,\ t]},\ \{w_i^*|_{[0,\ t]}\})$ is chosen that contains minimum collective energy.

The minimizing unknowns $(X_0^*,\ v^*|_{[0,\ t]},\ \{w_i^*|_{[0,\ t]}\})$ replaced in the system equation (1) yield the optimal state trajectory $X^*_{[0,\ t]}$. The subscript $[0,\ t]$ indicates that the optimization takes place on the interval $[0,\ t]$. The minimum-energy estimates at time t is defined as the final time value of the optimal state, $\hat{X}(t) := X^*_{[0,\ t]}(t)$.

In [21] the authors proposed an approximate solution to the above problem, an attitude filter called the geometric approximate minimum-energy (GAME) filter. In this work we consider a least squares analysis in order to evaluate how close to optimal the approximated solution of the GAME filter is.

V. MAIN RESULTS

This section contains a least squares analysis of the geometric approximate minimum-energy (GAME) filter that provides a mathematical expression of an upper bound on the (optimality) distance between the solution of the GAME filter and a minimum-energy (optimal) solution, although the minimum-energy filter is not expressed explicitly. This distance is quantifiable in simulations and is shown to be small in a comprehensive set of experiments involving different levels of initialization and measurement errors in Section VI, hence indicating that the GAME filter asymptotically acts like a minimum-energy filter. The term 'near-optimal' is sometimes used for filters with such performance characteristics [19], [20].

Recall the GAME filter [21]

$$\begin{aligned} &\dot{\hat{X}} = \hat{X}(u - Pl)_\times,\ \hat{X}(0) = \mathcal{X}, \\ &l = 2\,\mathrm{vex}(\sum_i \mathbb{P}_a(\hat{y}_i(\hat{y}_i - y_i)^\top R_i^{-1}),\ \hat{y} = \hat{X}^\top \mathring{y}. \end{aligned} \tag{18}$$

$$\begin{aligned} &\dot{P} = Q + \mathbb{P}_s(P(2u - Pl)_\times) - PSP + PAP, \\ &P(0) = (\mathrm{trace}(K_{X0})I - K_{X0})^{-1}, S = \sum_i (\hat{y}_i)_\times^\top R_i^{-1}(\hat{y}_i)_\times, \\ &A = \mathrm{trace}(C)I - C,\ C = \sum_i \mathbb{P}_s(R_i^{-1}(\hat{y} - y_i)\hat{y}_i^\top). \end{aligned} \tag{19}$$

We will state three lemmas that lead to the main results of this paper, Theorem 1, in which the distance to optimality of the GAME filter is shown to be bounded from above by a 'small' optimality gap $W(t)$ (23). Lemma 2 shows that the cost (17) can be rewritten with a new expression comprising the optimality gap $W(t)$ (23). The positivity of $W(t)$ is considered in Lemma 3.

Let us introduce some auxiliary variables first. Denote $K := P^{-1}$. The time derivative of K is given by

$$\dot{K} = -KQK + \mathbb{P}_s(K(2u - Pl)_\times) + S - A, \tag{20}$$

where the initial condition $K(0) = \mathrm{trace}(K_{X_0})I - K_{X_0}$ and the variables l, S and A are given from (18) and (19). This easily follows from $\dot{K} = P^{-1}\dot{P}P^{-1}$. The following lemma will be handy in our follow up calculations.

Lemma 1: Denote $G := \frac{1}{2}\mathrm{trace}(K)I - K$, then the following identities hold.

$$\begin{aligned} &\mathrm{trace}(G) = \frac{1}{2}\mathrm{trace}(K), K = \mathrm{trace}(G)I - G, \\ &\dot{G} = -\frac{1}{2}\mathrm{trace}(KQK)I + KQK + \mathbb{P}_s(G(2u - Pl)_\times) \\ &\qquad + \frac{1}{2}\mathrm{trace}(S)I - S - C, G(0) = K_{X_0}. \end{aligned} \tag{21}$$

Proof is straightforward using the definition of G and the Riccati (19).

The following lemma shows that the cost J_t (17) satisfies an equation that is comprised of a term depending on the

Preprint submitted to 2014 American Control Conference.
Received September 27, 2013.

difference between the current-time value of the state $X(t)$ and its estimated value $\hat{X}(t)$, as well as on an integral term which is an 'unavoidable optimal cost' and a remaining term $W(t)$ that is called the 'optimality gap' of the GAME filter.

Lemma 2: The cost (17) satisfies

$$J_t = \operatorname{trace}((I - E(t))G(t)) + \frac{1}{2}\int_0^t \Bigg(\sum_i \|y_i - \hat{y}_i\|^2_{R_i^{-1}} + \|v - 2B^\top \operatorname{vex} \mathbb{P}_a(E^\top G)\|^2 \Bigg) d\tau + W(t), \tag{22}$$

where the error is denoted by $E := \hat{X}^\top X$ and

$$W(t) := \int_0^t \Bigg(\frac{1}{2}\sum_i \|(\hat{X} - X)^\top \mathring{y}_i\|^2_{R_i^{-1}} - 2\|B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2 + \operatorname{trace}\Bigg[(I - \mathbb{P}_s(E)) \Bigg(\frac{1}{2}\operatorname{trace}(KQK)I - KQK + \frac{1}{2}\operatorname{trace}(S)I - S \Bigg) \Bigg] \Bigg) d\tau, \tag{23}$$

with the term S given from (19).

Proof: Consider the function

$$\mathcal{L} = \operatorname{trace}\Big[(I - \hat{X}^\top X)G \Big]. \tag{24}$$

The time derivative of Equation (24) is given by

$$\dot{\mathcal{L}} = \operatorname{trace}\Big[-(\dot{\hat{X}}^\top X)G - (\hat{X}^\top \dot{X})G + (I - \hat{X}^\top X)\dot{G} \Big]. \tag{25}$$

Substituting Equations (14) and (18) yields

$$\dot{\mathcal{L}} = \operatorname{trace}\Big[-(u - Pl)_\times^\top \hat{X}^\top X G - \hat{X}^\top X (u - Bv)_\times G + (I - \hat{X}^\top X)\dot{G} \Big]. \tag{26}$$

Using the trace property (7) and grouping the terms with u in Equation (26) yield

$$\dot{\mathcal{L}} = \operatorname{trace}\Big[-2\mathbb{P}_s(u_\times G)\mathbb{P}_s(E) - \mathbb{P}_s(E)\mathbb{P}_s(G(Pl)_\times) - \mathbb{P}_a(E)\mathbb{P}_a(G(Pl)_\times) + (Bv)_\times^\top \mathbb{P}_a(E^\top G) + (I - \mathbb{P}_s(E))\dot{G} \Big]. \tag{27}$$

Rewrite the term with v in a vector form using (3) and add and subtract the needed terms for completing the square of v.

$$\dot{\mathcal{L}} = 2v^\top B^\top \operatorname{vex}\mathbb{P}_a(E^\top G) \pm \frac{1}{2}v^\top v \pm 2\|\operatorname{vex}\mathbb{P}_a^\top(E^\top G)\|_{BB^\top} + \operatorname{trace}\Big[-2\mathbb{P}_s(u_\times G)\mathbb{P}_s(E) - \mathbb{P}_s(E)\mathbb{P}_s(G(Pl)_\times) - \mathbb{P}_a(E)\mathbb{P}_a(G(Pl)_\times) + (I - \mathbb{P}_s(E))\dot{G} \Big]. \tag{28}$$

Completing the square and replacing $\dot{G}$ from (1) yield

$$\dot{\mathcal{L}} = -\frac{1}{2}\|v - 2B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2 + \frac{1}{2}\|v\|^2 + 2\|B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2 + \operatorname{trace}\Bigg[-2\mathbb{P}_s(u_\times G)\mathbb{P}_s(E) - \mathbb{P}_a(E)\mathbb{P}_a(G(Pl)_\times) - \mathbb{P}_s(E)\mathbb{P}_s(G(Pl)_\times) + (I - \mathbb{P}_s(E))(-\frac{1}{2}\operatorname{trace}(KQK)I + KQK + \mathbb{P}_s(G(2u - Pl)_\times) + \frac{1}{2}\operatorname{trace}(S)I - S - C) \Bigg]. \tag{29}$$

Note that the first two terms in the trace part of the previous equation cancel the third term that replaced $\dot{G}$ multiplied by $\mathbb{P}_s(E)$. The third term that replaced $\dot{G}$ multiplied by I yields zero under the trace operator according to (7). Using identity (10), Lemma 1 and (18) we have

$$\mathbb{P}_a(G(Pl)_\times) = \frac{1}{2}((\operatorname{trace}(G)I - G)Pl)_\times = \frac{1}{2}l_\times = -\sum_i \mathbb{P}_a(R_i^{-1}(\hat{y}_i - y_i)\hat{y}_i^\top). \tag{30}$$

Consider a function $\phi(.)$ defined for symmetric matrices $H = H^\top \in \mathbb{R}^{3\times 3}$ that maps H to $\phi(H) = \frac{1}{2}\operatorname{trace}(H) - H$. We will use this function to simplify the notation in (29). Replacing the previous identity and also the equivalence of C from (19) into Equation (29) yield

$$\dot{\mathcal{L}} = 2\|B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2 - \frac{1}{2}\|v - 2B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2 + \frac{1}{2}\|v\|^2 + \operatorname{trace}\Bigg[\mathbb{P}_a(E)\sum_i \mathbb{P}_a(R_i^{-1}(\hat{y}_i - y_i)\hat{y}_i^\top) + \mathbb{P}_s(E)\sum_i \mathbb{P}_s(R_i^{-1}(\hat{y}_i - y_i)\hat{y}_i^\top) - \sum_i \mathbb{P}_s(R_i^{-1}(\hat{y}_i - y_i)\hat{y}_i^\top) + (I - \mathbb{P}_s(E))(-\phi(KQK) + \phi(S)) \Bigg]. \tag{31}$$

Note that the third term in the trace is equal to $-\sum_i R_i^{-1}(\hat{y}_i - y_i)\hat{y}_i^\top$ as the symmetric projection is redundant under the trace operator. Recalling the definitions $E = \hat{X}^\top X$ and $\hat{y}_i = \hat{X}^\top \mathring{y}$, the first two terms of the trace can be rewritten in the following form.

$$\operatorname{trace}[E\sum_i R_i^{-1}(\hat{y}_i - y_i)\hat{y}_i^\top] = \operatorname{trace}[\sum_i R_i^{-1}(\hat{y}_i - y_i)\mathring{y}_i^\top X]. \tag{32}$$

This expression substituted into (31) is grouped with the third term in the trace and the result is written in a vector inner product form. Also adding and subtracting the square norm

Preprint submitted to 2014 American Control Conference.
Received September 27, 2013.

of w_i yield

$$\dot{\mathcal{L}} = 2\|B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2 - \frac{1}{2}\|v - 2B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2 + \frac{1}{2}\|v\|^2 + \sum_i \left(\pm\frac{1}{2}\|w_i\|^2 - (\hat{y}_i - X^\top \mathring{y}_i)^\top R_i^{-1}(\hat{y}_i - y_i)\right) + \operatorname{trace}\left[(I - \mathbb{P}_s(E))(-\phi(KQK) + \phi(S))\right]. \tag{33}$$

Next we replace $y = X^\top \mathring{y} + D_i w_i$ and $R_i = D_i D_i^\top$ and complete the square of ω_i.

$$\dot{\mathcal{L}} = 2\|B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2 - \frac{1}{2}\|v - 2B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2 + \frac{1}{2}\sum_i \left(\|w_i\|^2 - \|y_i - \hat{y}_i\|^2_{R_i^{-1}} - \|\hat{y}_i - X^\top \mathring{y}_i\|^2_{R_i^{-1}}\right) + \frac{1}{2}\|v\|^2 + \operatorname{trace}\left[(I - \mathbb{P}_s(E))(-\phi(KQK) + \phi(S))\right]. \tag{34}$$

Integrating $\dot{\mathcal{L}}$ to obtain $\mathcal{L}(t) - \mathcal{L}(0) = \int_0^t \dot{\mathcal{L}} d\tau$, and replacing $\hat{X}(0) = \mathcal{X}$ yields the Lemma Equation (22) with the optimality gap W defined in (23) that completes the proof. ∎

Up to this stage it was shown that the cost (17) satisfies an equation that involves an 'unavoidable optimal cost' that doesn't depend on the error E and an 'optimality gap' $W(t)$. Provided that the term $W(t)$ is nonnegative, it can be considered as as an upper bound for the optimality performance of the GAME filter. In the following, the optimality gap W is expressed in a more familiar vector seminorm that facilitates proving it is nonnegative.

Lemma 3: The matrix

$$\Lambda = (I - E)(R_i^{-1} - E^{\frac{\top}{2}} R_i^{-1} E^{\frac{1}{2}})(I - E)^\top, \tag{35}$$

is nonnegative definite. Moreover, the mathematical expression for $W(t)$ (23) is equivalent to

$$W(t) = \int_0^t \left(\frac{1}{2}\sum_i \|\hat{y}_i\|^2_\Lambda + 2\|\operatorname{vex}\mathbb{P}_a(E^{\frac{1}{2}})\|^2_{KQK} - 2\|\operatorname{vex}\mathbb{P}_a(GE)\|^2_Q\right) d\tau, \tag{36}$$

yielding a nonnegative value if the matrix

$$Q - e_\times^\top Q e_\times \geq 0, \tag{37}$$

where the unit norm vector e is the axis of rotation of the error E.

Proof: From (1), it is straightforward to conclude that every rotation matrix has a square root $X^{\frac{1}{2}} := \exp(\frac{1}{2}\theta a_\times)$ such that $X = X^{\frac{1}{2}} X^{\frac{1}{2}}$. Using this fact, consider the function $\operatorname{trace}((I - X)\phi(H))$ where $H = H^\top \in \mathbb{R}^{3\times 3}$. Due to the invariance of the trace under cyclic permutations and from (4) we have

$$\operatorname{trace}((I - X)\phi(H)) = 2\operatorname{trace}(\mathbb{P}_a^\top(X^{\frac{1}{2}})\phi(H)\mathbb{P}_a(X^{\frac{1}{2}})) = 2\|\operatorname{vex}\mathbb{P}_a(X^{\frac{1}{2}})\|^2_H, \tag{38}$$

that is nonnegative if the matrix H is nonnegative.

Since the matrix R^{-1} is positive definite, it can be said that the matrix $E^{\frac{\top}{2}} R_i^{-1} E^{\frac{1}{2}}$ is nonnegative definite. Furthermore, recall that $E^{\frac{\top}{2}}$ is a rotation matrix and the eigenvalues of a rotation matrix occur in one of the forms

- Three eigenvalues equal to 1 (the rotation E equals the identity matrix I in this case).
- One eigenvalue equals to 1 and the other two are -1 (rotation by 180 degrees).
- One eigenvalue equals to 1 but the rest are complex conjugates of the form $\cos(\theta) \pm i\sin(\theta)$ (rotation through an angle of θ).

Therefore, it is straightforward to see that the real part of the eigenvalues of $E^{\frac{\top}{2}}$ are less than or equal to 1. Thus, $R_i - E^{\frac{\top}{2}} R_i^{-1} E^{\frac{1}{2}} \geq 0$. Similarly since $I - E \geq 0$ the matrix $\Lambda = (I - E)R_i - E^{\frac{\top}{2}} R_i^{-1} E^{\frac{1}{2}}(I - E)^\top$ is nonnegative definite.

Rewriting the trace part of (23), using (38), as a vector norm yields the optimality gap

$$W(t) = \int_0^t \left(\frac{1}{2}\sum_i \|(I - E)^\top \hat{y}_i\|^2_{R_i^{-1}} - 2\|\operatorname{vex}\mathbb{P}_a(E^\top G)\|^2_Q + 2\|\operatorname{vex}\mathbb{P}_a(E^{\frac{1}{2}})\|^2_{KQK - \hat{y}_\times^\top R_i^{-1}\hat{y}_\times}\right) d\tau. \tag{39}$$

Using the exponential formula (2), the term involving $E^{\frac{1}{2}}$ can be written as

$$2\|\operatorname{vex}\mathbb{P}_a(E^{\frac{1}{2}})\|^2_{KQK} = 2\sin^2(\frac{\theta}{2})\|Ke\|^2_Q = \frac{1}{2}(\sin^2(\theta) + (1 - \cos(\theta))^2)\|Ke\|^2_Q. \tag{40}$$

Recall that $G = \phi(K) = \frac{1}{2}\operatorname{trace}(K)I - K$. The exponential formula (2), the vector product formulas (8) and (9) and the weighted vector norm formula (12) can help rewriting the second term of (39) into

$$\begin{aligned} &-2\|\operatorname{vex}\mathbb{P}_a(GE)\|^2_Q = \\ &-2\|\operatorname{vex}\mathbb{P}_a(\sin(\theta)Ge_\times + (1 - \cos(\theta))Ge_\times^2)\|^2_Q = \\ &-2\|\operatorname{vex}\mathbb{P}_a(\sin(\theta)Ge_\times - (1 - \cos(\theta))Kee^T)\|^2_Q = \\ &-\frac{1}{2}\|\sin(\theta)Ke - (1 - \cos(\theta))(e)_\times Ke\|^2_Q, \end{aligned} \tag{41}$$

that using the triangle inequality yields

$$-2\|\operatorname{vex}\mathbb{P}_a(GE)\|^2_Q \geq -\frac{1}{2}\sin^2(\theta)\|Ke\|^2_Q - \frac{1}{2}(1 - \cos(\theta))^2\|e_\times Ke\|^2_Q \tag{42}$$

Equations (40) and (42) and condition (37) yield that

$$2\|\operatorname{vex}\mathbb{P}_a(E^{\frac{1}{2}})\|^2_{KQK} - 2\|\operatorname{vex}\mathbb{P}_a(GE)\|^2_Q \geq \frac{1}{2}(1 - \cos(\theta))^2\|Ke\|^2_{Q - e_\times^\top Q e_\times} \geq 0. \tag{43}$$

Preprint submitted to 2014 American Control Conference.
Received September 27, 2013.

Next, rewriting the second term involving $\hat{y}_i$ yields

$$\begin{aligned}&2\|\operatorname{vex}\mathbb{P}_a(E^{\frac{1}{2}})\|^2_{\hat{y}_\times^\top R_i^{-1}\hat{y}_\times} = \\ &2((\hat{y}_i)_\times \operatorname{vex}\mathbb{P}_a(E^{\frac{1}{2}}))^\top R_i^{-1}((\hat{y}_i)_\times \operatorname{vex}\mathbb{P}_a(E^{\frac{1}{2}})) = \\ &2(\mathbb{P}_a(E^{\frac{1}{2}})\hat{y}_i)^\top R_i^{-1}(\mathbb{P}_a(E^{\frac{1}{2}}) = \\ &\frac{1}{2}\hat{y}_i^\top((I-E)(E^{\frac{\top}{2}}R_i^{-1}E^{\frac{1}{2}})(I-E)^\top)\hat{y}.\end{aligned} \tag{44}$$

Combining the $\hat{y}_i$ terms in (39) using (44) concludes that (36) holds. Furthermore, the optimality gap (36) is nonnegative due to (43) and the fact that $\Lambda \geq 0$. ■

Lemma 3 proves that the optimality gap W is nonnegative given that the condition (37) is satisfied. Note that this condition is straightforward for instance if the matrix Q is a multiple of the identity matrix by observing that from (8) $I - e_\times^\top e_\times = ee^\top$.

Theorem 1: Consider the system (14) and the cost (17). Given some measurements $\{y_i|_{[0,\ t]}\}$ and their associated inputs $u|_{[0,\ t]}$, assume that unique solutions $\hat{X}$ and $P(t)$ to (18) and (19) exist on $[0, t]$. Moreover, assuming that condition (37) holds, then the filter (18) and (19) yields a near-optimal estimate $\hat{X}(t)$ of the state $X(t)$ in the sense that for each time t there exists a hypothesized trajectory X_{ht} with the final value $X_{ht}(t) = \hat{X}(t)$ and $J_t \leq J_t^* + W(t)$, where J_t is the cost for X_{ht}, J_t^* denotes the minimum-energy value for the cost (17) and $W(t)$ (36) is a bound on the optimality distance between the two trajectories X_{ht} and X_t^*, the latter denoting a minimum-energy trajectory corresponding to J_t^*.

Proof: Lemma 2 states that

$$\begin{aligned}J_t = \operatorname{trace}((I-E(t))G(t)) + \frac{1}{2}\int_0^t \Bigg(\sum_i \|y_i - \hat{y}_i\|^2_{R_i^{-1}} \\ + \|v - 2B^\top \operatorname{vex}\mathbb{P}_a(E^\top G)\|^2\Big)\, d\tau + W(t),\end{aligned} \tag{45}$$

where $E = \hat{X}^\top X$. From (38) it is evident that the first term in the previous expression is nonnegative and furthermore since $W \geq 0$, the cost function J_t fulfills the inequality

$$J_t \geq \frac{1}{2}\int_0^t \sum_i \|y_i - \hat{y}_i\|^2_{R_i^{-1}} d\tau. \tag{46}$$

The right hand side of Equation (46) is independent of any specific choice of the unknown arguments of the cost (17), X_0, $v|_{[0,\ t]}$ and $\{w_i|_{[0,\ t]}\}$, and depends only on the measured data $\{y_i|_{[0,\ t]}\}$ and the filter estimates. Thus, the right hand side of Equation (46) is also a lower bound for the minimum J_t^* of the cost (17), i.e.

$$J_t^* \geq \frac{1}{2}\int_0^t \sum_i \|y_i - \hat{y}_i\|^2_{R_i^{-1}} d\tau. \tag{47}$$

Consider a hypothesis $X_{ht} : [0,\ t] \longrightarrow \mathrm{SO}(3)$ for the true trajectory of the system generated by

$$\dot{X}_{ht} = X_{ht}(u - 2Q\operatorname{vex}\mathbb{P}_a(X_{ht}^\top \hat{X} G))_\times, \tag{48}$$

with fixed *final* condition $X_{ht}(t) := \hat{X}(t)$ where $\hat{X}$ and G are solutions of the proposed filter through (21). It is straightforward to show (by integrating in reverse time) that (48) has a unique initial state $X_{ht}(0)$ that produces the final condition $X_{ht}(t) = \hat{X}(t)$. Define the signals $(w_i)_{ht}$: $[0,\ t] \longrightarrow \mathbb{R}^3$ by

$$(w_i)_{ht} := D_i^{-1}(y_i - \hat{y}_i), \tag{49}$$

and the signal $v_{ht} : [0,\ t] \longrightarrow \mathbb{R}^3$ by

$$v_{ht} := 2B^\top \operatorname{vex}\mathbb{P}_a(X_{ht}^\top \hat{X} G) \tag{50}$$

Equations (48) and (49) show that $X_{ht}(0)$, $v_{ht}|_{[0,\ t]}$ and $\{(w_i)_{ht}|_{[0,\ t]}\}$ together with $u|_{[0,\ t]}$ and $\{y_i\}|_{[0,\ t]}$ satisfy the system equations (14).

Recalling Lemma 2 the functional cost J_t of X_{ht} is

$$J_t = \frac{1}{2}\int_0^t \sum_i \|y_i - \hat{y}_i\|^2_{R_i^{-1}} d\tau + W(t) \leq J_t^* + W(t). \tag{51}$$

This completes the proof. ■

As was mentioned before condition (37) is straightforward by choosing the design parameter Q in the form of a multiple of the identity matrix.

A key contribution of Theorem 1 lies in providing a bound $W(t)$ given by (36) for evaluating the performance of the GAME filter. An interesting question is to check how small the optimality gap is. Note that if the matrix R is a multiple of the identity matrix, then the matrix Λ (35) is equal to zero that leads to a smaller optimality gap. Furthermore, from (36) it is clear that this bound is decreasing with the tracking error $E = \hat{X}^\top X$. Thus, once the initial transient of the filter is complete, and for moderate modeling error, it is to be expected that the filter will perform qualitatively as well as an optimal filter. This is further investigated in the following simulations.

VI. Simulations

In this section we compute the value of the optimality gap (36) in simulations with Monte-Carlo experiments of 100 repeats in order to test how close to optimal the GAME filter is. The GAME filter is simulated using the identity matrix as both the initial rotation estimate and the initial gain. A sinusoidal input $\Omega = [0.2\sin(\frac{\pi}{3}t)\ \ -\cos(\frac{\pi}{3}t)\ \ 2\cos(\frac{\pi}{3}t)]$ drives the true trajectory X. We further assume that two orthogonal unit reference vectors are available.

To consider a challenging filtering scenario, we have simulated relatively high levels of initialization and measurement errors. The coefficient matrix B is chosen so that the signal Bv has a standard deviation of 60 degrees per 'second'. The system is initialized with a rotation of 120 degrees. Gaussian zero mean measurement noise signals w_i with unit standard deviations are considered for which the coefficient matrices D_i are chosen so that the signals $D_i w_i$ have standard deviations of 90 degrees. Figure 2 is the plot of $W(t)$. Note that after a short transient period the average value of $W(t)$ reaches a plateau that indicates that once the GAME filter converges it acts like a minimum-energy filter apart from the initially accumulated optimality error bound.

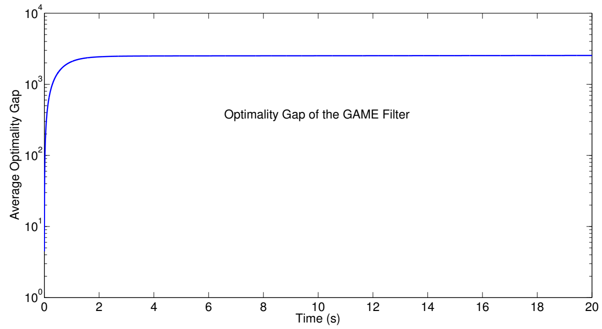

Fig. 2. The average of the bound on the optimality performance of the GAME filter ($W(t)$) over 100 repeats plotted against time.

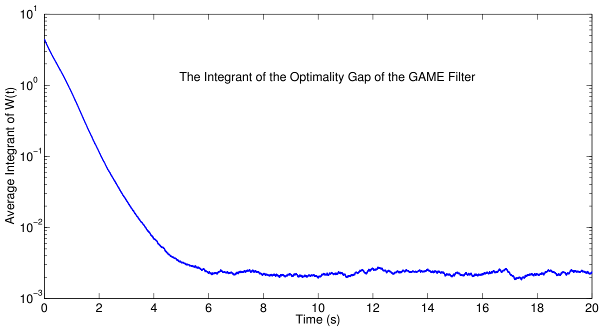

Fig. 3. The average of the integrand of $W(t)$ over 100 repeats plotted against time.

This is more clear in Figure 3 where the average of the integrand of $W(t)$ throughout the 100 experiments is plotted against time.

VII. CONCLUSIONS

In this work we showed that the recently proposed geometric approximate minimum-energy (GAME) attitude filter is in fact near-optimal meaning that the cost attained by the GAME filter is close to the minimum-energy cost. This further supports the superior performance of the GAME filter against other attitude filtering results that was demonstrated in [7] using simulations. Another contribution of this work is the analytical expression of the optimality gap of the GAME filter that can be computed to analyze its performance in applications where validation data is available. The authors are not aware of any similar performance measure available for any other attitude filter. As a future research direction, it would be interesting to find out if a similar optimality gap can be derived for the well-known multiplicative extended Kalman filter (MEKF) and to compare that against the GAME filter.

REFERENCES

[1] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Journal of Basic Engineering, Transactions of the ASME*, vol. 82, no. 1, pp. 35–45, 1960.

[2] B. F. La Scala, R. R. Bitmead, and M. R. James, " Conditions for stability of the Extended Kalman Filter and their application to the frequency tracking problem," *Mathematics of Control, Signals, and Systems (MCSS)*, vol. 8, no. 1, pp. 1–26, March 1995.

[3] S. J. Julier and J. K. Uhlmann, "Reduced sigma point filters for the propagation of means and covariances through nonlinear transformations," in *Proceedings of the American Control Conference*, vol. 2, 2002, pp. 887–892.

[4] M. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking," *IEEE Transactions on Signal Processing*, vol. 50, no. 2, pp. 174–188, 2002.

[5] F. Markley, "Attitude error representations for Kalman filtering," *Journal of guidance, control, and dynamics*, vol. 26, no. 2, pp. 311–317, 2003.

[6] J. L. Crassidis, F. L. Markley, and Y. Cheng, "Survey of nonlinear attitude estimation methods," *Journal of Guidance Control and Dynamics*, vol. 30, pp. 12–28, 2007.

[7] M. Zamani, "Deterministic attitude and pose filtering, an embedded lie groups approach," Ph.D. dissertation, The Australian National University, 2013.

[8] M. Zamani, J. Trumpf, and R. Mahony, "A second order minimum-energy filter on the special orthogonal group," in *The 2012 American Control Conference (ACC)*, 2012, pp. 1895–1900.

[9] R. E. Mortensen, "Maximum-likelihood recursive nonlinear filtering," *Journal of Optimization Theory and Applications*, vol. 2, no. 6, pp. 386–394, 1968.

[10] O. B. Hijab, "Minimum energy estimation," Ph.D. dissertation, University of California, Berkeley, 1980.

[11] S. Bonnabel, "Left-invariant extended Kalman filter and attitude estimation," in *Proceedings of the IEEE Conference on Decision and Control*, 2007, pp. 1027–1032.

[12] Bonnabel, S. and Martin, P. and Salaun, E., "Invariant Extended Kalman Filter: theory and application to a velocity-aided attitude estimation problem," in *Proceedings of the 48th IEEE Conference on Decision and Control*, 2009, pp. 1297–1304.

[13] P. Martin, E. Salaün, *et al.*, "Generalized multiplicative extended kalman filter for aided attitude and heading reference system," in *Proceedings of 2010 AIAA Guidance, Navigation, and Control Conference*, 2010.

[14] J. Crassidis and F. Markley, "Unscented filtering for spacecraft attitude estimation," *Journal of Guidance Control and Dynamics*, vol. 26, no. 4, pp. 536–542, 2003.

[15] R. Mahony, T. Hamel, and J. -M. Pflimlin, "Nonlinear complementary filters on the special orthogonal group," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1218, 2008.

[16] S. I. Marcus, "Algebraic and geometric methods in nonlinear filtering," *SIAM Journal on Control and Optimization*, vol. 22, no. 6, pp. 817–844, 1984.

[17] A. Sanyal, "Optimal Attitude Estimation and Filtering Without Using Local Coordinates Part I: Uncontrolled and Deterministic Attitude Dynamics," in *American Control Conference*, 2006.

[18] A. Sanyal, T. Lee, M. Leok, and N. McClamroch, "Global optimal attitude estimation using uncertainty ellipsoids," *Systems & Control Letters*, vol. 57, no. 3, pp. 236–245, 2008.

[19] P. Coote, J. Trumpf, R. Mahony, and J. C. Willems, "Near-optimal deterministic filtering on the unit circle," in *Prooceedings of the 48th IEEE Conference on Decision and Control*, 2009, pp. 5490–5495.

[20] M. Zamani, J. Trumpf, and R. Mahony, "Near-Optimal deterministic filtering on the rotation group," *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 1411–1414, 2011.

[21] M. Zamani, J. Trumpf, and R. Mahony, "A second order minimum-energy filter on the special orthogonal group," in *In Proceedings of the 2012 American Control Conference (ACC), to appear. Accepted for publication 28 January*, 2012.